Planar Ramsey Numbers of Four Cycles Versus Wheels

Chen Yaojun[1], Miao Zhengke[2] and Zhou Guofei[1]
1. Department of Mathematics, Nanjing University, Nanjing 210093, China
2. Department of Mathematics, Jiangsu Normal University

Abstract

For two given graphs G and H the planar Ramsey number $PR(G,H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G, or its complement contains a copy of H. In this paper, we first characterize some structural properties of C_4-free planar graphs, and then we determine all planar Ramsey numbers $PR(C_4, W_n)$, for $n \geq 3$.

1 Introduction

In this paper, all graphs are simple. Given two graphs G and H, the Ramsey number $R(G,H)$ is the smallest integer n such that every graph F on n vertices contains a copy of G, or its complement contains a copy of H. The determination of Ramsey numbers is an extremely difficult problem. In this paper, we are interested in planar Ramsey numbers. For two given graphs G and H the planar Ramsey number $PR(G,H)$ is the smallest integer n such that every planar graph F on n vertices either contains a copy of G, or its complement contains a copy of H. The concept of planar Ramsey number was introduced by Walker [5] in 1969 and by Steinberg and Tovey [4] in 1993, independently.

For planar Ramsey number, all pairs of complete graphs was determined in [4]. Gorgol and Rucinski [3] determined all pairs of cycles. By combining computer search with some theoretical results, A. Dudek and A. Rucinski [2] compute most of the planar Ramsey numbers $PR(G_1, G_2)$, where each of G_1 and G_2 is a complete graph, a cycle or a complete graph without one edge. In [6], Zhou et. al. determined $PR(C_3, W_n)$ for $n \geq 3$.

In this paper, we first characterize some structural properties of C_4-free planar graphs, and then we determine all planar Ramsey numbers $PR(C_4, W_n)$, for $n \geq 3$.

2 Some concepts and notations

Let $G = (V(G), E(G))$ be a graph and G^c the complement of G. We define $\varepsilon(G) = |E(G)|$.

Let v be a vertex in G, the neighborhood of v, denoted by $N_G(v)$, is the vertex set consisting of the vertices which are adjacent to v. We define $N_G[v] = N_G(v) \cup \{v\}$. We denote by $d_G(v) = |N_G(v)|$ the degree of v in G. The maximum and minimum degrees in G, will be denoted by $\Delta(G)$ and $\delta(G)$ respectively.

Let $U \subseteq V(G)$, denote by $G[U]$ the subgraph induced by U in G. The independence number, the connectivity and the minimum degree of G, are denoted by $\alpha(G)$, $k(G)$ and $\delta(G)$ respectively.

Let v be a vertex in G and H be a subgraph in G, we denote by $v+H$ the graph in which every vertex of H is adjacent to v. A *wheel* $W_n = \{x\} + C_n$ is a graph of order $n+1$, where x is called the hub of the wheel, C_n is a cycle of length n, and x is adjacent to each vertices of C_n.

A graph G of order n is said to be Hamiltonian if it contains an n-cycle; and G is said to be pancyclic if G contains cycles of length k, for all $k = 3, 4, \cdots, n$.

A planar graph which is embedded in a plane is called a plane graph, a face of length k in G is called a k-face, whose boundary has exactly k edges, denote by $F(G), F_k(G)$ the set of faces and the set of k-faces of G, respectively. Let f_k be the number of k-faces in G.

Let C be a cycle of a plane graph G, we call C a separating cycle of G if both the inside and outside of C have at least one vertex.

Let G be a plane graph, we denote by $\Gamma(G)$ the set of edges which are not covered by any triangle. The cardinality of $\Gamma(G)$ is denoted by $\tau(G)$.

Let G be a plane graph. We can construct a new graph G^* from G as follows: the vertex set of G^* consists of all the faces of lengths at least 5, and for each pair of vertices $f, g \in V(G^*)$, f and g are adjacent if and only if f and g have exactly one vertex or have exactly one edge in common. For convenience, we call G^* the vertex-edge-dual of G.

If G has e_1 vertices of degree d_1, e_2 vertices of degree d_2,$\cdots$, e_k vertices of degree d_k, we will denote the degree sequence of G by $d_1^{e_1} d_2^{e_2} \cdots d_k^{e_k}$, where $d_1 \leq d_2 \leq \cdots \leq d_k$.

We define $\delta(n, C_4) = \max\{\delta(G) | G \text{ is a } C_4 \text{ free planar graph }\}$; We denote by $M(n, C_4)$ the maximum number of edges among all C_4-free planar graphs. In [7], Zhou and Chen determined all the values of $M(n, C_4)$ for $n \geq 30$.

Our main result is the following two theorems:

Theorem 2.1 *Let* $A = \{30, 36, 39, 42\} \cup \{k | k \geq 44\}$ *and* $B = \{k | 10 \leq k \leq 29\} \cup \{31, 32, 33, 34, 35, 37, 38, 40, 41, 43\}$ *be two integer sets. Then*
(i) $\delta(n, C_4) = 4$*, for each* $n \in A$*;*
(ii) $\delta(n, C_4) = 3$*, for each* $n \in B$*;*
(iii) $\delta(n, C_4) = 2$*, for each* $5 \leq n \leq 9$*.*

Theorem 2.2 *The planar Ramsey numbers of* C_4 *versus* W_n *is as follows:*

$$PR(C_4, W_n) = \begin{cases} 10, & \text{if } n = 3; \\ 9, & \text{if } n = 6; \\ n+4, & \text{if } n \in \{k | 7 \leq k \leq 25\} \cup \{27, 28, 29, 30, 31, 33, 34, 36, 37, 39\}; \\ n+5, & \text{if } n \in \{4, 5, 26, 32, 35, 38\} \cup \{k | k \geq 40\}. \end{cases}$$

3 The min-max degrees in C_4-free planar graphs

The following result can be implied by the famous Euler's Formula on plane graphs, and we omit the proof here.

Theorem 3.1 *Let G be a C_4-free plane graph, then $\varepsilon(G) = \frac{15}{7}(n-2) - \frac{2}{7}\tau(G) - \frac{3}{7}f_6 - \frac{6}{7}f_7 - \cdots - \frac{3(r-5)}{7}f_r \leq \frac{15}{7}(n-2)$ (where r is the maximum length of faces in G).*

Corollary 3.1 *If G is a C_4-free planar graph of order n, then*
(i) $\delta(G) \leq 4$.
(ii) $\delta(G) \leq 3$, *if* $n \leq 29$.

Proof. (i) Let G be a C_4-free planar graph of order n. Suppose on the contrary that $\delta(G) \geq 5$, which implies that the number of edges of G is at least $\frac{5}{2}n > \frac{15}{7}(n-2)$, this contradicts Theorem 3.1.

(ii) If $\delta(G) \geq 4$, by (i), we have $\delta(G) = 4$, thus the number of edges is $2n \leq \frac{15}{7}(n-2)$, this implies that $n \geq 30$, a contradiction. $\square$

Lemma 3.1 *[7] $M(n, C_4) = \lfloor 15(n-2)/7 \rfloor - \mu$ for $n \geq 30$, where $\mu = 1$ if $n \equiv 3(\text{mod}7)$ or $n = 32, 33, 37$, and $\mu = 0$ otherwise.*

By using the program PLANTRI by Brinkmann and McKay [8], we have checked the fact of the follwing three facts.

Fact 3.1 *There are all together 3 non-isomorphic triangulations of planar graphs on 16 vertices with minimum degree $\delta = 5$ (Figure 1).*

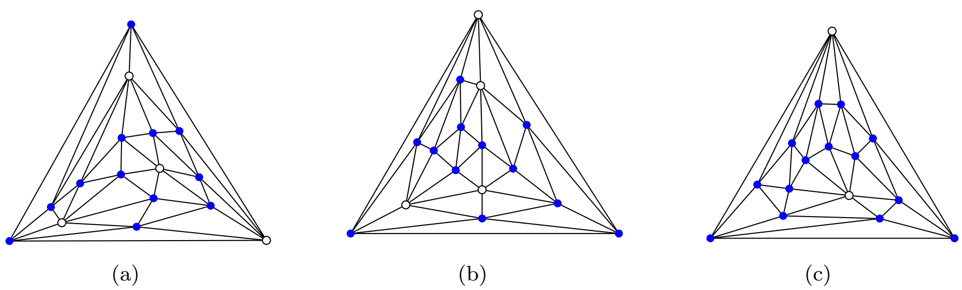


Figure 1: 3 non-isomorphic triangulations of 16 vertices with $\delta = 5$.

Fact 3.2 *There are all together 4 non-isomorphic triangulations of planar graphs on 17 vertices with $\delta = 5$ (Figure 2).*

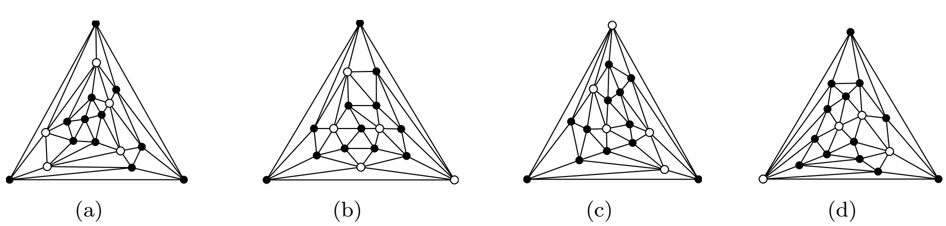


Figure 2: All non-isomorphic triangulations of 17 vertices with $\delta = 5$.

Fact 3.3 *Let G be a triangulation on 18 vertices with minimum degree 5, and let T be the set of vertices whose degree is at least 6 in G, then there is no C_5 in the subgraph induced by T.*

Lemma 3.2 *Let G be a C_4-free plane graph. Let H be the subgraph induced by $\Gamma(G)$.*
(i) *If G is 4-regular, then either $\tau(G) = 0$ or $\tau(G) \geq 5$.*
(ii) *If G is 4-regular and $\tau(G) = 5$, then H is a 5-cycle.*
(iii) *If $\delta(G) \geq 4$, and there is an edge $e = uv \in \Gamma(G)$ such that $d_G(u) = 4$, then there is an edge $f = uw$ ($w \neq v$) such that $f \in \Gamma(G)$.*

Proof. Suppose that G is 4-regular and $\tau(G) \neq 0$, let uv be an arbitrary edge in $\Gamma(G)$, then there exists two faces $g_1, g_2 \in F(G) - F_3(G)$ such that g_1, g_2 have an edge uv in common (see Figure 3). Since G is 4-regular, $f_1 \neq f_2$ and $f_3 \neq f_4$. Furthermore, since G is C_4-free, at least one of f_1 and f_2 are non-triangles, so at least one of vx_1 and vx_2 are not covered by triangles. For the same reason as above, we see that at least one of ux_3 and ux_4 are not covered by triangles. This implies H has at least 4 vertices and that both $d_H(u) \geq 2$ and $d_H(v) \geq 2$, and hence $\delta(H) \geq 2$. If $\tau(G) = 4$, then H will contain a 4-cycle, a contradiction. So we have $\tau(G) \geq 5$. This complete the proof of (i).

By the above arguments, we see that $\delta(H) \geq 2$ and it is obvious that (ii) and (iii) holds.

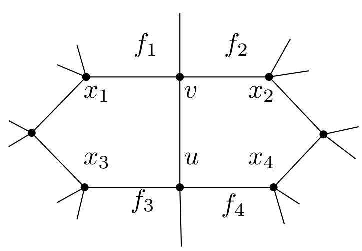


Figure 3: $\tau(G) \geq 5$

Lemma 3.3 *Let G be a 4-regular C_4-free plane graph with $\tau(G) = 5, f_6 \leq 1, f_7 = f_8 = \cdots = 0$, then $\Gamma(G)$ does not induce a separating 5-cycle in G.*

Proof. Let H be the subgraph induced by $\Gamma(G)$. By the proof of Lemma 3.2, we know that $\delta(H) \geq 2$. Since G is C_4-free, H must be a 5-cycle C in G. Suppose on the contrary that C is a separating 5-cycle.

Claim. For each vertex v in $V(C)$, the two edges which are adjacent to v and which are not on C must be either outside or inside C, but not both.

Proof of Claim. Let e_1, e_2 be the two edges which are incident with v and which are not on C. Suppose, without loss of generality, that e_1 is inside C, and e_2 is outside C (Figure 4). Since e_1 is covered by a triangle in G, and e_3, e_4 are not covered by any triangle, this is impossible since G is 4-regular and C_4-free. $\square$

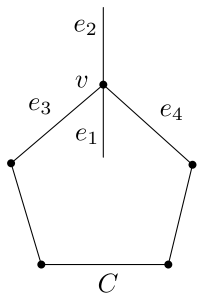


Figure 4: A forbidden structure

Let t_1 and t_2 be the number of edges that are incident with $V(C)$ and belong to the inside of C and the outside of C respectively. Since G is 4-regular, by Claim we see that $t_1 + t_2 = 10$, and both t_1 and t_2 are even integers. Without loss of generality, we assume that $t_1 \leq 4$. Suppose first that $t_1 = 2$ (Figure 5). In this case we can embed all the vertices of inside C to outside C, this contradicts that C is a separating cycle.

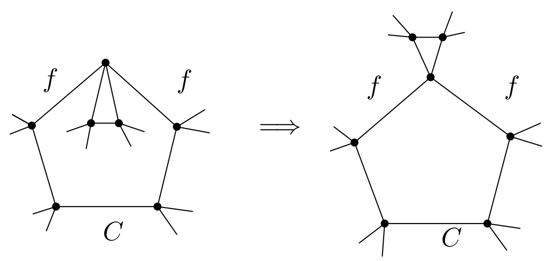


Figure 5: Re-embedding of G

Suppose next that $t_1 = 4$, let u, v be the two vertices on C such that the edges which are not on C and are incident with u, v are inside C. If u, v are adjacent on C, then we can also re-embed all the vertices inside C to outside C (Figure 6), this contradicts again that C is a separating cycle.

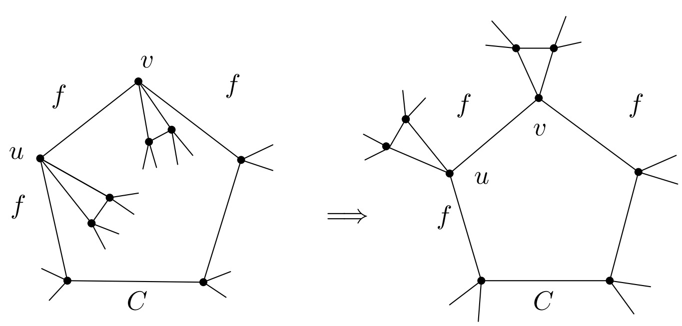


Figure 6: Another re-embedding of G

So we assume that u, v are not adjacent on C (Figure 7). In this case, consider the face f inside C which are incident with u_1u_2. Since G is C_4-free, $u_3 \neq u_4$. Hence u, u_1, u_2, v, u_3, u_4 are all on the

boundary of f, which implies that the length of f is at least 6, so f must be a 6-face because of the initial hypothesis that for each $k \geq 7$, $f_k(G) = 0$. This implies that u_3 and u_4 are adjacent. Consider the face g which is incident with w and inside C, since $f_6 \leq 1$ and $\{w, u, u_5, u_6, v\}$ is on the boundary of g, this implies that u_5 and u_6 are adjacent, but now $u_3u_4u_5u_6u_3$ is a C_4 in G, which contradicts the initial hypothesis that G is C_4-free. □

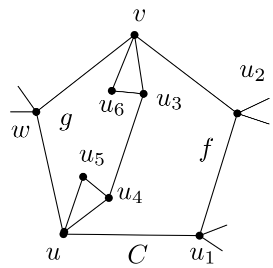


Figure 7: An forbidden structure of G.

3.1 Proof of Theorem 2.1

Proof. (i) By Corollary 3.1, it suffices to show that for each $n \in A$, there is a C_4-free planar graph G of order n which has minimum degree 4.

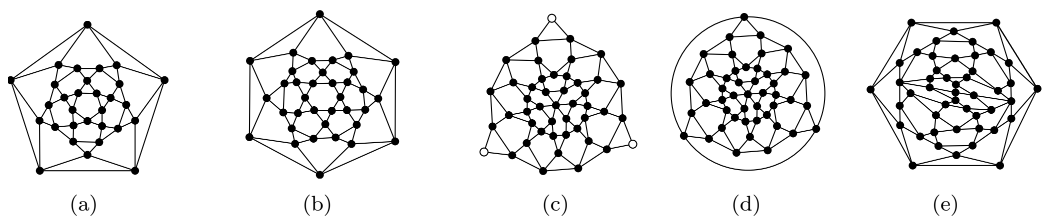


Figure 8: Five C_4-free planar graphs with $\delta = 4$.

Figure 8 illustrates five C_4-free planar graphs with minimum degree 4, for $n = 30, 36, 44, 46, 47$ respectively (where the three white vertices of graph (c) are identified). Note that each planar graph in Figure 8(b),(d),(e) has at least one 6-face.

We begin to construct a new C_4-free planar graph with n vertices and with minimum degree $\delta(G) = 4$ from one of the graphs illustrated in Figure 8 (b), (d), (e).

Take one k-face f with $k \geq 6$, we construct a new planar graph G^* by operation (A) which is illustrated in Figure 9. In this operation, we find two vertices u, v which has distance 3 on f, then split u and v into two vertices u_1, u_2 and v_1, v_2 respectively. Finally we add a new vertex (the white vertex) inside f, and add edges from it to u_1, u_2, v_1, v_2 respectively. The resulting graph G^* is C_4-free and with minimum degree $\delta = 4$ and with three more vertices. We can see that G^* still has a k-face with $k \geq 6$ (one of the face incident with u_1u_2 or v_1v_2). So we can take operation (A) again on the k-face (with $k \geq 6$) on G^*, and therefore get a new C_4-free planar graph with minimum degree $\delta = 4$ and with three more vertices

than G^*.

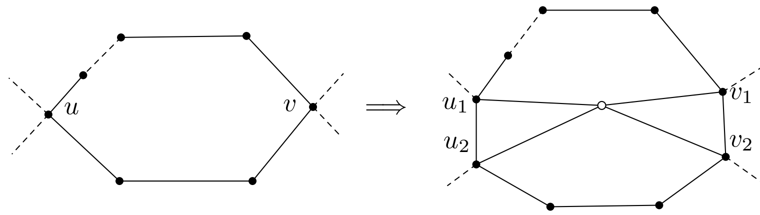


Figure 9: Operation (A)

In a result, if we take operation (A) recursively on graph (b) in Figure 8, we can construct C_4-free planar graph with $36+3t_1$ vertices and with minimum degree $\delta=4$, where $t_1 \geq 0$. Similarly, if we take operation (A) recursively on graph (d) and (e) in Figure 8 respectively, we can get graphs with $46+3t_2$ and $47+3t_3$ vertices, where $t_2, t_3 \geq 0$. This complete the proof of (i) in Theorem 2.1.

(ii) Let G be a C_4-free planar graph of order n and with $\delta(G)=\delta(n, C_4)$. If $31 \leq n \leq 33$, then by Lemma 3.1, we have $\varepsilon(G) \leq 2n-1$, this implies that $\delta(n, C_4) \leq 3$. If $n=34, 35, 37, 38$, then by Lemma 3.1 again, we have $M(n, C_4)=2n$ for $n=34, 35, 37, 38$, this implies that $\delta(G) \leq 4$. If $\delta(G)=4$, then G is 4-regular and $\varepsilon(G)=M(n, C_4)=2n$. By Theorem 3.1, $\varepsilon(G)=M(34, C_4)=68$ if and only if $\tau(G)=2$ and $f_6=\cdots=f_r=0$; $\varepsilon(G)=M(35, C_4)=70$ if and only if $\tau(G)=1$, $f_6=1$ and $f_7=\cdots=f_r=0$; $\varepsilon(G)=M(37, C_4)=74$ if and only if $\tau(G)=2$, $f_6=1$ and $f_7=\cdots=f_r=0$; $\varepsilon(G)=M(38, C_4)=76$ if and only if $\tau(G)=1$, $f_7=1$ and $f_6=f_8=\cdots=f_r=0$, or $\tau(G)=4$ and $f_6=\cdots=f_r=0$, or $\tau(G)=1$, $f_6=2$ and $f_7=\cdots=f_r=0$. But each of the above cases contradicts the facts of Lemma 3.2.

If $n=40$, suppose on the contrary that there is a C_4-free planar graph of order $n=40$ with $\delta(G) \geq 4$. By Theorem 3.1, we get that $40 \leq \varepsilon(G) \leq 81$.

Suppose first that $\varepsilon(G)=80$, then G is 4-regular, and further more, there are only three possibilities to consider: (a) $\tau(G)=2, f_7=1$ and $f_6=f_8=\cdots=f_r=0$; (b) $\tau(G)=2, f_6=2$ and $f_7=f_8=\cdots=f_r=0$; (c) $\tau(G)=5$ and $f_6=f_7=\cdots=f_r=0$. By Lemma 3.2, the first two possibilities can not happen.

So we assume that (c) holds. By Euler's formula, G has 17 pentagons and 25 triangles. Since $\tau(G)=5$ and by Lemma 3.2, $\Gamma(G)$ induces a 5-cycle C in G. By Lemma 3.3, C is a 5-face. Consider the vertex-edge-dual G^* of G, since G is C_4-free, G^* is a triangulation of 17 vertices with degree sequence $5^{12}6^5$; and furthermore, since $\Gamma(G)$ induces a 5-face, G^* has the property that there is a vertex of degree 5 which is adjacent to every vertex of degree 6. By checking the graphs in Figure 2, none of them has the above property, a contradiction. So in the following, we suppose that $\varepsilon(G)=81$.

By Theorem 3.1, this is possible only if $\tau(G)=0, f_6=1$, and $f_7=f_8=\cdots=0$. In this case G has 15 pentagons, 27 triangles and one 6-face. By the definition of vertex-edge-dual G^* of G, we see that G^* is a triangulation on 16 vertices with $\delta(G^*) \geq 5$. Since $\delta(G)=4, \tau(G)=0$ and $\varepsilon(G)=81$, the degree sequence of G is exactly $4^{39}6^1$. Let f be the 6-face and v be the vertex of degree 6 in G, let f_1, f_2, f_3 be the nontriangle faces which are incident to v. If f is not adjacent to v, then by the

rule of the construction of G^*, G^* have exactly four vertices of degree 6 (corresponding to f, f_1, f_2, f_3) and three of which (corresponding to f_1, f_2, f_3) form a triangle in G^*; If f is adjacent to v, then G^* have exactly two vertices of degree 6 and two vertices of degree 7. By Fact 3.1, there are all together 3 non-isomorphic triangulations on 16 vertices with minimum degree 5, and none of them has the above property, a contradiction.

Therefore, we have the conclusion that $\delta(40, C_4) \leq 3$.

If $n = 41$, suppose on the contrary that there is a C_4-free planar graph of order $n = 41$ with $\delta(G) \geq 4$. By Theorem 3.1, we have that $82 \leq \varepsilon(G) \leq 83$. If $\varepsilon(G) = 82$, then G is 4-regular, and by Theorem 3.1, there are four possibilities to consider:

(1) $\tau(G) = 1, f_6 = f_7 = 1, f_8 = f_9 = \cdots = 0$;

(2) $\tau(G) = 1, f_6 = 3, f_7 = f_8 = \cdots = 0$;

(3) $\tau(G) = 1, f_8 = 1, f_6 = f_7 = f_9 = \cdots = 0$;

(4) $\tau(G) = 4, f_6 = 1, f_7 = f_8 = \cdots = 0$.

But all these cases contradicts Lemma 3.2. So we have that $\varepsilon(G) = 83$. By Theorem 3.1, this can happen only if $\tau(G) = 2, f_6 = f_7 = \cdots = 0$. Furthermore, since $\delta(G) \geq 4$, the degree sequence of G is either $4^{40}6^1$ or $4^{39}5^2$. Assume first that the degree sequence of G is $4^{40}6^1$. If there is an edge $e \in \Gamma(G)$ (say $e = uv$) such that $d_G(u) = d_G(v) = 4$, by Lemma 3.2 (iii), we have that $\tau(G) \geq 3$, which contradicts that $\tau(G) = 2$; If the two edges $e, f \in \Gamma(G)$ satisfy that $e = wu, f = wv$ with $d_G(u) = d_G(v) = 4$ and $d_G(w) = 6$, by Lemma 3.2 (iii) again, we have that $\tau(G) \geq 4$, which is a contradiction too. So we assume that the degree sequence of G is $4^{39}5^2$. Let u, v in G such that $d_G(u) = d_G(v) = 5$ and let $\Gamma(G) = \{e, f\}$. By Lemma 3.2, it suffices to consider the case that e, f have a vertex in common and are incident with u, v respectively (Figure 10).

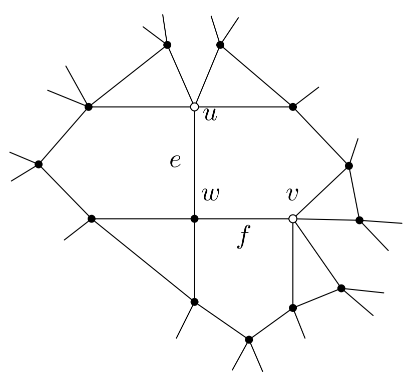


Figure 10: Local structure of G.

By Euler's Formula, G has 27 triangles and 17 pentagons. Consider the vertex-edge-dual G^* of G. It is a triangulation on 17 vertices with degree sequence $5^{14}6^3$, this is impossible since any triangulation on 17 vertices has $3 \times 17 - 6 = 45$ edges.

Therefore, we have the conclusion that $\delta(41, C_4) \leq 3$.

If $n = 43$, suppose on the contrary that there is a C_4-free planar graph of order $n = 43$ with $\delta(G) = 4$. By Theorem 3.1, we have $86 \leq \varepsilon(G) \leq 87$.

If $\varepsilon(G) = 86$, then G is 4-regular since $\delta(G) = 4$. By Theorem 3.1, there are four possibilities to consider:

(1) $\tau(G) = 2, f_8 = 1, f_6 = f_7 = f_9 = \cdots = 0$;

(2) $\tau(G) = 2, f_6 = 3, f_7 = f_8 = \cdots = 0$;

(3) $\tau(G) = 2, f_6 = f_7 = 1, f_8 = f_9 = \cdots = 0$;

(4) $\tau(G) = 5, f_6 = 1, f_7 = f_8 \cdots = 0$;

By Lemma 3.2, the first three cases can not happen. So we assume that $\tau(G) = 5, f_6 = 1, f_7 = f_8 \cdots = 0$. In this case, G is 4-regular, with 27 triangles, 17 pentagons and one hexagon.

Let $\Gamma(G) = \{e_1, e_2, e_3, e_4, e_5\}$. By Lemma 3.2 and 3.3, $\Gamma(G)$ induces a 5-face in G. Consider the vertex-edge-dual G^* of G, it is a triangulation on 18 vertices with degree sequence $5^{12}6^6$ or $5^{13}6^4 7^1$. Furthermore, G^* has the additional property: let T be the vertex set consisting of all vertices with degrees at least 6 in G^*, then there is a five cycle in the subgraph induced by T in G^*. By checking the graphs in Fact 3.3, none of them has that property, a contradiction.

Now we assume $\varepsilon(G) = 87$. By Theorem 3.1, we shall only consider the following three cases:

(1) $\tau(G) = 0, f_6 = 2, f_7 = f_8 \cdots = 0$;

(2) $\tau(G) = 0, f_7 = 1, f_6 = f_8 = f_9 = \cdots = 0$;

(3) $\tau(G) = 3, f_6 = f_7 = \cdots = 0$.

For case (1), by Euler's formula, we have $f_5 = 15, f_6 = 2$. Since $\delta(G) = 4$ and $\varepsilon(G) = 87$, the degree sequence of G is $4^{42}6^1$ or $4^{41}5^2$. Since G is C_4-free and $\tau(G) = 0$, there is no vertex of degree 5 in G, so the degree sequence of G is $4^{42}6^1$. Consider the vertex-edge-dual G^* of G, it is a triangulation on 17 vertices; Furthermore, it has at least 3 vertices of degree at least 6, and three of them form a triangle in G^*. By checking the graphs in Fact 3.2, we see that none of which has the above property, a contradiction.

For case (2), a similar argument as in case (i) will deduce a contradiction.

For case (3), by Euler's formula, we have $f_5 = 18$. Since $\delta(G) = 4$ and $\varepsilon(G) = 87$, the degree sequence of G is $4^{42}6^1$ or $4^{41}5^2$. Since $\tau(G) = 3$, let $\Gamma(G) = \{e_1, e_2, e_3\}$. We shall consider the following four cases:

Case 1. $\Gamma(G)$ forms a matching in G.

Since there are at most two vertices of degrees at least 5, there is an edge in $\Gamma(G)$ (say e_1) so that the degrees of both endpoints of e_1 are four. By Lemma 3.2, we have $\tau(G) \geq 5$, which contradicts that $\tau(G) = 3$.

Case 2. $\Gamma(G)$ induces two disjoint paths.

If the degree sequence of G is $4^{42}6^1$, then there must be an edge in $\Gamma(G)$ such that each endpoint of which has degree 4 in G, this implies by Lemma 3.2 that $\tau(G) \geq 4$, which contradicts that $\tau(G) = 3$.

If the degree sequence of G is $4^{41}5^2$, let $P_1 = u_1u_2u_3$, $P_2 = v_1v_2$ be the two disjoint paths induced by $\Gamma(G)$ respectively, and let u, v be the two vertices of degrees 5 in G. If $d_G(u_2) = 4$, then at least two vertices of $\{u_1, u_3, v_1, v_2\}$ have degrees 4 in G, this implies by Lemma 3.2 that $\tau(G) \geq 5$, which

contradicts that $\tau(G) = 3$. If $d_G(u_2) = 5$, then at east one vertex of v_1, v_2 has degree 4 in G, this implies by Lemma 3.2 that $\tau(G) \geq 4$, which contradicts again that $\tau(G) = 3$.

Case 3. $\Gamma(G)$ induces a path of length 3 in G.

Let $P = u_1u_2u_3u_4$ be the path induced by $\Gamma(G)$ in G. If $d_G(u_1) = 4$ or $d_G(u_4) = 4$, then $\tau(G) \geq 4$ by Lemma 3.2, which contradicts that $\tau(G) = 3$. This implies that the degree sequence of G is exactly $4^{41}5^2$, and that $d_G(u_1) = d_G(u_4) = 5$. Then G must have one of the following structure (Figure 11 (a) or (b)). Now we consider the vertex-edge-dual G^* of G, note that G has exactly 18 pentagons and no more faces of length at least 6, we can see that in both cases G^* are triangulations on 18 vertices with degree sequence $5^{14}6^4$, but this impossible since G^* is a triangulation on 18 vertices.

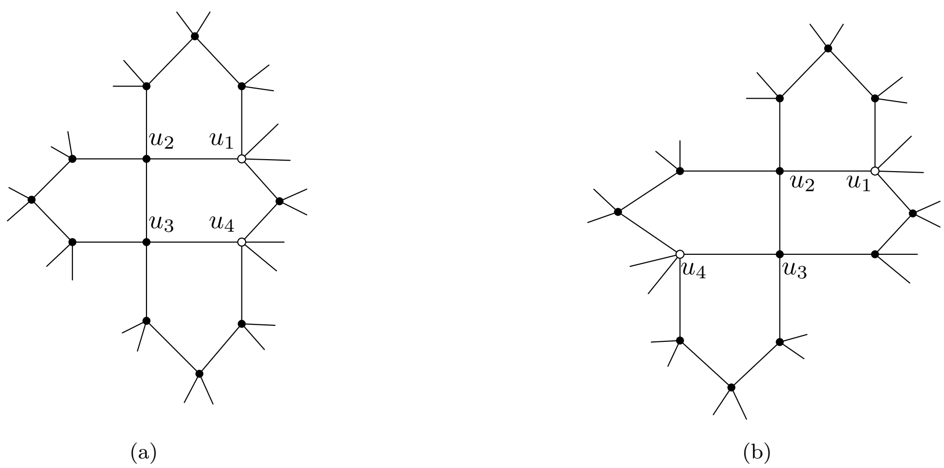


Figure 11: Local structure of G.

Case 4. $\Gamma(G)$ induces a 3-cycle in G.

In this case $\Gamma(G)$ must induces a separating triangle in G. By a similar argument as in the proof of Lemma 3.3 will deduce a contradiction.

Therefore, we have the conclusion that for each $n \in B = \{31, 32, 33, 34, 35, 37, 38, 40, 41, 43\}$, $\delta(n, C_4) \leq 3$.

Now it remains to show that for each $n \in B = \{31, 32, 33, 34, 35, 37, 38, 40, 41, 43\}$, there is a C_4-free planar graph G with $\delta(G) = 3$. We begin with the graph shown in Figure 8 (a), it is a C_4-free 4-regular planar graph on 30 vertices. Each time we take one vertex v with degree 4 in G, and make Operation (B) as shown in Figure 12 (where f and g are faces of length at least 5). In this operation, the vertex v is split to two vertices v_1 and v_2, then add an edge between v_1 and v_2. In this way, we get a new C_4-free planar graph with $\delta = 3$ with one more vertex. If we make the Operation (B) $n - 30$ times (note that $n \leq 43$, each time we can always find a vertex of degree 4 in the new graph), we finally get a C_4-free planar graph on n vertices with $\delta = 3$.

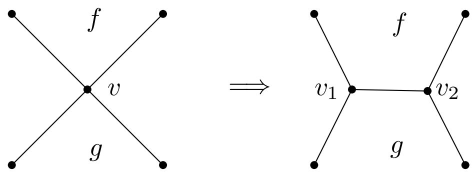


Figure 12: Operation (B).

If $10 \leq n \leq 29$, by corollary 3.1, $\delta(n, C_4) \leq 3$. So it suffices to show the existence of C_4-free planar graph on n vertices with minimum degree 3. We begin with the C_4-free planar graph G on 10 vertices with minimum degree 3 (Figure 13). Each time we take one of the following operations, and finally we can construct a C_4-free planar graphs on n vertices with minimum degree 3, where $10 \leq n \leq 29$.

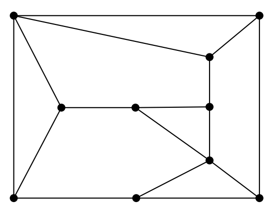


Figure 13: A C_4-free planar graph on 10 vertices with $\delta = 3$.

(1) Operation (B), as illustrated in Figure 12;

(2) The reverse operation of (B);

(3) Operation (C): take one edge e which are the common edge of two faces of lengths at least 6, split e in to three edges and add two more chordal edges between that two faces (as illustrated in Figure 14).

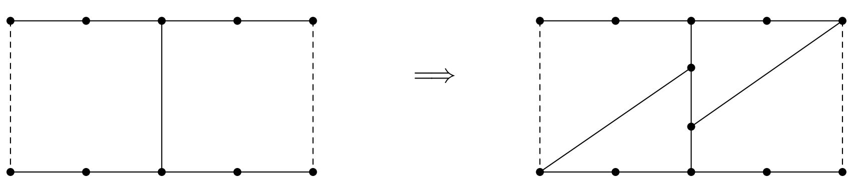


Figure 14: Operation (C).

(iii) If $n = 9$, suppose on the contrary that $\delta(9, C_4) = 3$, then there is a C_4-free planar graph G on 9 vertices with $\delta(G) = 3$. By Theorem 3.1, we have $\varepsilon(G) \leq 15$. Since $\delta(G) = 3$, we have $\tau(G) \neq 0$, which implies that $\varepsilon(G) \leq 14$, and the equality holds if and only if $\tau(G) = 2$ and $f_6 = 1$ in Theorem 3.1. Note that the degree sequence of G must be $3^8 4^1$, this implies that $\tau(G) \geq 4$, since each vertex of degree 3 is adjacent of at least one edge in $\tau(G)$. This is contradicts the fact that $\tau(G) = 2$. So we have $\delta(9, C_4) \leq 2$.

If $n = 8$, suppose on the contrary that $\delta(8, C_4) = 3$, then there is a C_4-free planar graph G on 8 vertices with $\delta(G) = 3$. By Theorem 3.1, we have $\varepsilon(G) \leq 12$. Since $\delta(G) = 3$, we have that $\varepsilon(G) = 12$, so the degree sequence of G must be 3^8, this implies that $\tau(G) \geq 4$, since each vertex of degree 3 is

adjacent of at least one edge in $\tau(G)$. Hence by Theorem 3.1 we have $\varepsilon(G) \leq \frac{15}{7}(8-2) - \frac{2}{7}4 < 12$. This is contradicts that $\varepsilon(G) = 12$. So we have $\delta(8, C_4) \leq 2$.

If $n = 7$, suppose on the contrary that $\delta(7, C_4) = 3$, then there is a C_4-free planar graph G on 7 vertices with $\delta(G) = 3$. By Theorem 3.1, we have $\varepsilon(G) \leq 10$. Since $\delta(G) = 3$, we have that $\varepsilon(G) = 11$, a contradiction. So we have $\delta(7, C_4) \leq 2$.

If $n \leq 6$, suppose on the contrary that $\delta(n, C_4) = 3$, then there is a C_4-free planar graph G on n vertices with $\delta(G) = 3$. On the one hand, since $\delta(G) = 3$, we have $\varepsilon(G) \geq \frac{3n}{2}$; On the other hand, by Theorem 3.1 we have $\varepsilon(G) \leq \frac{15}{7}(n-2)$, this is impossible since $n \leq 6$. So we have $\delta(6, C_4) \leq 2$ and $\delta(5, C_4) \leq 2$.

Since C_n is a C_4-free planar graph with minimum degree 2, we therefore conclude that $\delta(n, C_4) = 2$ for $5 \leq n \leq 9$. □

4 Proof of Theorem 2.2

In this section we begin to consider planar Ramsey numbers of C_4 versus wheels. The following two Lemmas are well known.

Lemma 4.1 *(Dirac) If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

Lemma 4.2 *(Chvátal-Erdös) If $\alpha(G) \leq k(G)$, then G is Hamiltonian.*

In [1], Brandt proved that

Lemma 4.3 *Every non-bipartite graph of order n with more than $(n-1)^2/4 + 1$ edges contains cycles of every length between 3 and the length of a longest cycle.*

Lemma 4.4 *Let G be a C_4-free planar graph, then its independence number $\alpha(G^c) \leq 3$.*

Proof. If $\alpha(G^c) \geq 4$, then G contains a K_4, and hence contains a C_4, which contradicts the initial hypothesis. □

Lemma 4.5 *Let G be a C_4-free planar graph with order $n \geq 6$ and $k(G^c) \leq 2$, then there exists two vertices x, y which separates some vertex z from the rest in G^c, and further more, $G - \{x, y, z\}$ contains no path of length 2 in G.*

Proof. (a) Since $k(G^c) \leq 2$, there exists two vertices x, y which separates U_1 from the rest U_2 in G^c. Then each vertex of U_1 is adjacent to every vertex of U_2 in G. If both $|U_1| \geq 2$ and $|U_2| \geq 2$, then G will contain a C_4, a contradiction.

(b) Note that z is adjacent to each vertex of $V(G) - \{x, y, z\}$ in G. If $G - \{x, y, z\}$ contains a path of length 2 in G, then there will be a C_4 in G, a contradiction. □

Lemma 4.6 *(I. Gorgol and A. Rucinski [3]) $PR(C_4, C_3) = PR(C_3, C_4) = PR(C_4, C_5) = 7$, and $PR(C_4, C_n) = n + 1$ for $n \geq 6$.*

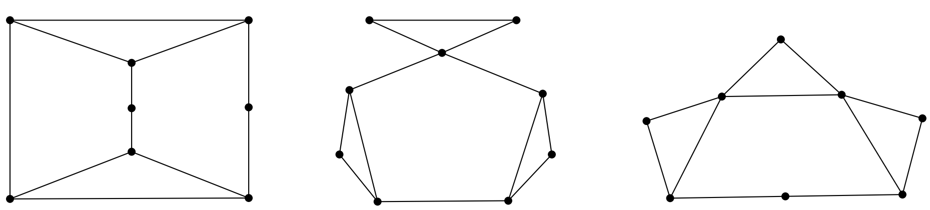
Lemma 4.7 *Let G be a C_4-free planar graph on $n \geq 7$ vertices, then G^c contains cycles of lengths from 3 to $n-1$.*

Proof. Let G be a C_4-free planar graph on $n \geq 7$ vertices, by Lemma 4.6, G^c contains a C_{n-1}. Furthermore, G^c is not a bipartite graph, otherwise, there will be at least one partite set with cardinality at least 4 since $n \geq 7$, which will induce a complete graph in G, and hence G will contain a 4-cycle, a contradiction. By Theorem 3.1, the number of edges of G is at most $\frac{15}{7}(n-2)$. So the number of edges of G^c is at least $\binom{n}{2} - \frac{15}{7}(n-2) > \frac{(n-1)^2}{4} + 1$, for $n \geq 7$. By Lemma 4.3, G^c contains cycles of lengths from 3 to $n-1$. $\square$

Bielak and Gorgol proved that $PR(C_4, K_4) = 10$, since $K_4 = W_3$, so $PR(C_4, W_3) = 10$.

In Figure 15 we illustrate three C_4-free planar graphs which contain no W_4, W_5 and W_6 respectively, this implies that $PR(C_3, W_4) \geq 9,\ PR(C_3, W_5) \geq 10,\ PR(C_3, W_6) \geq 9$. As a matter of fact, by using a program "Planram" due to Andrzej Dudek [9], we can easily check the following planar ramsey numbers:

Lemma 4.8 $PR(C_4, W_4) = 9,\ PR(C_4, W_5) = 10,\ PR(C_4, W_6) = 9.$



(a) (b) (c)

Figure 15: C_4-free W_n-free planar graphs for $n = 4, 5, 6$ respectively.

Lemma 4.9 *Let G be a C_4-free planar graph on 11 vertices, then G^c contains a W_7.*

Proof. By Corollary 3.1, we know that $\delta(G) \leq 3$. Let v be a vertex in G such that $d_G(v) = \delta(G)$, let H be the subgraph induced by the vertex set $V - N_G[v]$ in G, then H is a C_4-free planar graph on $10 - d_G(v)$ vertices. It suffices to show that H^c contains a C_7.

Case 1. $\delta(G) \leq 2$.

In this case we have $|V(H)| \geq 8$. Let $U \subseteq V(H)$ such that $|U| = 8$, then $G[U]$ is a C_4-free planar graph on 8 vertices, by Lemma 4.6, $G^c[U]$ contains a C_7, and hence H^c contains a C_7 too.

Case 2. $\delta(G) = 3$.

In this case we have $|V(H)| = 7$. It suffices to show that H^c is Hamiltonian.

Let t be the number of vertices which have degrees 3 in G. Since G is C_4-free, for every vertex u which has odd degree, there must be at least one edge in $\Gamma(G)$ which is incident with u. This implies that $\tau(G) \geq \frac{t}{2}$. By Theorem 3.1, we have $\frac{1}{2}(3t + 4(11-t)) = \varepsilon(G) \leq \frac{15}{7}(11-2) - \frac{2}{7} \cdot \frac{t}{2}$, this implies that $t \geq 8$.

If $t = 8$, assume that $\Delta(G) \geq 5$, then $\varepsilon(G) \geq 19$, but by Theorem 3.1, $\varepsilon(G) \leq \frac{15}{7}(11-2) - \frac{2}{7}\tau(G) < 19$, a contradiction. So we assume that $\Delta(G) = 4$, this means that the degree sequence of G is $3^8 4^3$, hence $\varepsilon(G) = 18$. On the other hand, since $\tau(G) \geq 4$, it is obvious by Theorem 3.1 that $\varepsilon(G) = \frac{15}{7}(11-2) - \frac{2}{7}\tau(G) - \frac{3}{7}f_6 - \frac{6}{7}f_7 - \cdots - \frac{3(r-5)}{7}f_r \neq 18$ (where r is the maximum length of face in G), a contradiction.

If $t = 9$, then $\varepsilon(G) \geq 18$ and $\tau(G) \geq 5$. On the other hand, by Theorem 3.1, $\varepsilon(G) \leq \frac{15}{7}(11-2) - \frac{2}{7}\tau(G) < 18$, a contradiction.

So the only possible case is that $t = 10$. If $\Delta(G) \geq 5$, then $\varepsilon(G) \geq 18$, On the other hand, by Theorem 3.1, $\varepsilon(G) \leq \frac{15}{7}(11-2) - \frac{2}{7}\tau(G) < 18$, a contradiction.

So we assume that $\Delta(G) = 4$, and thus the degree sequence of G is $3^{10}4^1$. We choose v such that $d_G(v) = 3$ and the only vertex of degree 4 belongs to $N_G(v)$.

If $k(H^c) \geq 3$, then by Lemmas 4.2 and 4.4, H^c is Hamiltonian. So in the following, we may assume that $k(H^c) \leq 2$. By Lemma 4.5, there exists two vertices x, y which separates z from the rest in H^c, which implies that $d_{H^c}(z) \leq 2$, and thus $4 \geq d_G(z) \geq d_H(z) \geq 4$, so $d_G(z) = 4$. By the choice of v, we know that $z \in N_G(v)$, a contradiction. $\square$

Note that if G is a planar graph of order N with $\delta = \delta(G)$, then G^c can not contain a $W_{N-\delta}$, by Theorem 2.1, we get the following lower bounds of planar Ramsey numbers:

Corollary 4.1

$$PR(C_4, W_n) \geq \begin{cases} n+4, & \text{if } n \in \{k|7 \leq k \leq 25\} \cup \{27, 28, 29, 30, 31, 33, 34, 36, 37, 39\}; \\ n+5, & \text{if } n \in \{26, 32, 35, 38\} \cup \{k|k \geq 40\}. \end{cases}$$

Lemma 4.10 *Let G be a C_4-free planar graph on N ($N \geq 12$) vertices and let $n = N - \delta(N, C_4) - 1$, then G^c contains W_n and W_{n-1}.*

Proof. By Corollary 3.1, we know that $\delta(G) \leq \delta(N, C_4) \leq 4$. Since $n = N - \delta(N, C_4) - 1$ and $N \geq 12$, we get that $n \geq 7$. Let v be a vertex in G such that $d_G(v) = \delta(G)$, let H be the subgraph induced by the vertex set $V - N_G[v]$ in G, then H is a C_4-free planar graph on $N - d_G(v) - 1 \geq n$ vertices.

Case 1. $\delta(G) \leq \delta(N, C_4) - 1$.

In this case we have $|V(H)| \geq n+1$, by Lemma 4.7, H^c contains cycles of lengths from 3 to $|V(H)|-1 \geq n$, let C_{n-1} and C_n be the cycles of lengths $n-1$ and n respectively, hence $v + C_{n-1}$ and $v + C_n$ are W_{n-1} and W_n in G^c respectively.

Case 2. $\delta(G) = \delta(N, C_4)$.

In this case we have $|V(H)| = n$.

By Lemma 4.6, H^c contains a C_{n-1}, and hence $v + C_{n-1}$ is a W_{n-1} in H^c. Next, we shall show that H^c contains a W_n.

If $k(H^c) \geq 3$, then by Lemmas 4.2 and 4.4, H^c is Hamiltonian. Let C be a Hamiltonian cycle in H^c, then $v + C$ is a W_n in G^c. So in the following, we may assume that $k(H^c) \leq 2$. By Lemma 4.5, there exists two vertices x, y which separates z from the rest. Let $U = V(H) - \{x, y, z\}$.

Note that z is adjacent to each vertex of U in G.

If $\delta(G) = \delta(N, C_4) = 4$, the number of edges of G is at least $\frac{1}{2}((n-3)+4(n+4)) \geq \frac{15}{7}(n+3)+1 = \frac{15}{7}(|V(G)|-2)+1$, which contradicts Theorem 3.1.

Since $N \geq 12$, we assume that $\delta(G) = \delta(N, C_4) = 3$ by Theorem 2.1. In this case $|U| = N - 7$.

Since G is C_4-free and z is adjacent to each vertex of U in G, each vertex of $N_G(v) \cup \{x, y\}$ can be adjacent to at most one vertex in $V(H) - \{x, y, z\}$ in G. So the edges between $N_G[v] \cup \{x, y, z\}$ and U is at most $|U| + 5$; On the other hand, since $\delta(G) = 3$ and their is no path of length 2 in the subgraph of G induced by U by Lemma 4.5, the number of edges between $N_G[v] \cup \{x, y, z\}$ and U is at least $3|U| - 2[\frac{|U|}{2}]$ (where $[x]$ denotes the maximum integer which is at most x). So we have that $3|U| - 2[\frac{|U|}{2}] \leq |U| + 5$, which is impossible since $N \geq 12$. □

Combining Theorem 2.1, corollary 4.1 and Lemmas 4.9,4.8 and 4.10, we finally prove Theorem 2.2.

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